Evolving Apollonian networks with small-world scale-free topologies

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We propose two types of evolving networks: evolutionary Apollonian networks (EANs) and general deterministic Apollonian networks (GDANs), established by simple iteration algorithms. We investigate the two networks by both simulation and theoretical prediction. Analytical results show that both networks follow power-law degree distributions, with distribution exponents continuously tuned from 2 to 3. The accurate expression of clustering coefficient is also given for both networks. Moreover, the investigation of the average path length of EAN and the diameter of GDAN reveals that these two types of networks possess small-world feature. In addition, we study the collective synchronization behavior on some limitations of the EAN.

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I. INTRODUCTION

In the last few years, complex networks have attracted a growing interest from a wide circle of researchers [1–4], with particular focus on the following three properties: power-law degree distribution [5], small average path length (APL), and high clustering coefficient [6]. The reason for this boom is that complex networks describe many systems in nature and society which share the above mentioned three characteristics.

In order to mimic the real-world systems, a wide variety of models have been proposed [1–4]. The first successful attempt is the Watts and Strogatz's (WS) small-world network model [6], which started an avalanche of research on the models of small-world networks and the WS model [7–11]. Another well-known model is Barabási and Albert's (BA) elegant scale-free network model [5], which has attracted an exceptional amount of attention within the physics community. In addition to analytic studies of the BA model and research of its extensions or modifications [12–14], many authors have developed a considerable number of other models and mechanisms that may represent processes more realistically taking place in real-world networks [15–23]. Modeling complex networks with small-world and scale-free properties is still an important issue.

Recently, based on the well-known Apollonian packing, Andrade *et al.* introduced Apollonian networks [24] which were also proposed by Doye and Massen in Ref. [25] simultaneously. Apollonian networks belong to a deterministic growing type of networks, which have drawn much attention from the scientific community [26–35]. The effects of the Apollonian networks on several dynamical models have been intensively studied, including the Ising model and a magnetic model [24,36–38]. Doye and Massen adopted an extension [25] of the two-dimensional model to investigate energy landscape networks [39,40]. Zhang *et al.* proposed a minimal iterative algorithm for constructing high dimensional networks and studied their structural properties [41]. In Ref. [42] Zhou *et al.* proposed a simple rule that generates random two-dimensional Apollonian networks, which are generalized by Zhang *et al.* to high dimension [43]. The deterministic Apollonian networks (DANs) [24,25,41] and random Apollonian networks (RANs) [42,43] may provide valuable insight into the real-life networks.

In this paper, first we propose a general scenario for constructing evolutionary Apollonian networks (EANs) controlled by a parameter q. The EAN can also result from Apollonian packing and unifies the DAN and RAN to the same framework, i.e., the DAN and RAN are special cases of EAN. Then, we present a general deterministic Apollonian network (GDAN) model governed by a single parameter m. Both the EAN and GDAN have a power-law degree distribution, a very large clustering coefficient, and a small intervertex separation. The degree exponent of EAN and GDAN is changeable between 2 and 3. Moreover, we introduce an algorithm for the DAN and RAN which can realize the construction of our networks. In the end, through the eigenvalue spectrum of the Laplacian matrix, the synchronizability on some limiting cases of the EAN is discussed.

II. BRIEF INTRODUCTION TO DETERMINISTIC AND RANDOM APOLLONIAN NETWORKS

We first introduce Apollonian packing (see Fig. 1 for an example of two dimensions), which dates back to the ancient Greek mathematician Apollonius of Perga. The classic two-dimensional Apollonian packing is constructed as follows. Initially three mutually touching disks are inscribed inside a circular space which is to be filled. The interstices of the initial disks and circle are curvilinear triangles to be filled. We denote this initial configuration by generation $t=0$. Then in the first generation $t=1$, four disks are inscribed, each touching all the sides of the corresponding curvilinear triangle. The process is repeated indefinitely for all the new curvilinear triangles. In the limit of infinite generations, we obtain the well-known two-dimensional Apollonian packing. The translation from Apollonian packing construction to

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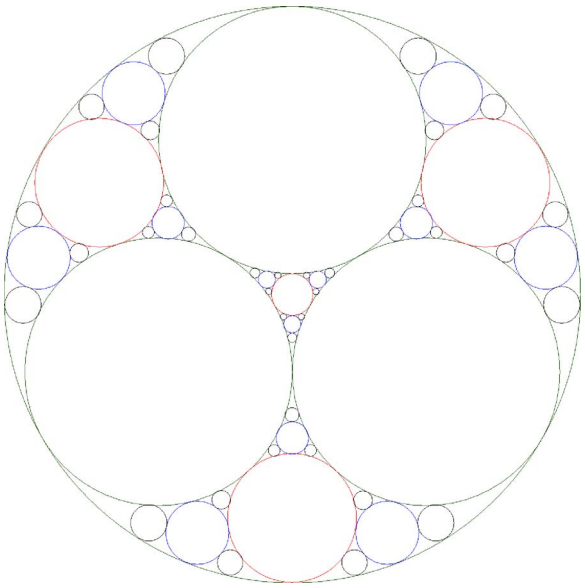

FIG. 1. (Color online) A two-dimensional Apollonian packing of disks.

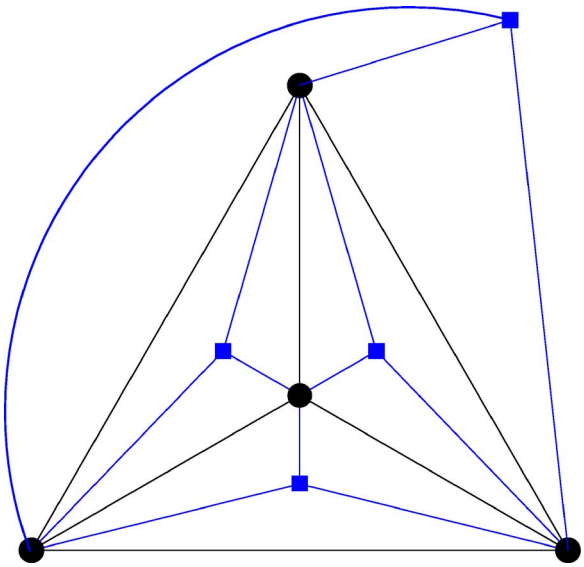

FIG. 2. (Color online) Illustration of the two-dimensional deterministic Apollonian networks, showing the first two steps of iterative process.

Apollonian network generation is quite straightforward. Let the nodes (vertices) of the network correspond to the disks and make two nodes connect if the corresponding disks are tangent [24,25]. Figure 2 shows a network based on the two-dimensional Apollonian packing. The two-dimensional Apollonian network can be easily generalized to high dimensions (d-dimensional, $d \geqslant 2$) [25,41] associated with other self-similar packings [44]. The d-dimensional Apollonian packings start with $d+1$ mutually touching d-dimensional hyperspheres that is enclosed within and touching a larger d-dimensional hypersphere, with $d+2$ curvilinear d-dimensional simplices (d simplices) as their interstices, which are to be filled in successive generations. If each d hypersphere corresponds to a node and nodes are connected if the corresponding d hyperspheres are in contact, then d-dimensional Apollonian networks are established. In every generation of the d-dimensional Apollonian packings, if we add only one d hypersphere inside a randomly selected interstice, then we get high dimensional random Apollonian packings, based on which high dimensional random Apollonian networks are constructed [42,43].

III. THE ITERATIVE ALGORITHMS FOR THE NETWORKS

Before introducing the algorithms we give the following definitions on a graph. The term "size" refers to the number of edges in a graph. The number of nodes in the graph is called its "order." When two nodes of a graph are connected by an edge, these nodes are said to be "adjacent," and the edge is said to join them. A "complete graph" is a graph in which all nodes are adjacent to one another. Thus, in a complete graph, every possible edge is present. The complete graph with d graph nodes is denoted as K_d (also referred to in the literature as a "d clique;" see Ref. [45]). Two graphs are "isomorphic" when the nodes of one can be relabeled to match the nodes of the other in a way that preserves adjacency. Hence all d cliques are isomorphic to one another.

A. Evolutionary Apollonian networks

Now we introduce the evolving Apollonian networks. First we give a new packing method for high-dimensional (d-dimensional, $d \geqslant 2$) Apollonian packings. The initial configuration is the same as the deterministic Apollonian packings. Then in each subsequent generation, each d simplex is filled with probability q. In a special case $q=1$, it is reduced to the classic deterministic Apollonian packings. if q approaches but is not equal to 0, it coincides with the random Apollonian packings described in Refs. [42,43]. The EAN is derived from this new packing: nodes represent d hyperspheres and edges correspond to contact relationship. Figure 2 shows the network growing process for the special case of $d=2$ and $q=1$.

In the construction process of the new high-dimensional Apollonian packings, for each new d hypersphere, $d+1$ new interstices are created. When building networks, it is equivalent that for each new added node, $d+1$ new d simplices are generated, which may create new nodes in the subsequent generations. According to this, we can introduce a general algorithm to create an EAN, using which one can write a computer program conveniently to simulate the networks and study their properties.

The d-dimensional EANs after t generations are denoted by $A(d,t)$, $d \geqslant 2$, $t \geqslant 0$. Then at step t, the d-dimensional EAN is constructed as follows. For $t=0$, $A(d,0)$ is a complete graph K_{d+2} [or $(d+2)$ clique]. For $t \geqslant 1$, $A(d,t)$ is obtained from $A(d,t-1)$. For each of the existing subgraphs of $A(d,t-1)$ that is isomorphic to a $(d+1)$ clique and has never generated a node before [we call them active $(d+1)$ cliques], with probability q, a new node is created and connected to all the nodes of this subgraph. The growing process is repeated

until the network reaches a desired order. When $q=1$, the networks are exactly the same as the DAN [24,25,41]. If $q<1$, the networks grow randomly. Especially, as q approaches zero and does not equal zero, the networks are reduced to the RAN studied in detail in Refs. [42,43]. See Ref. [46] for interpretation.

Next we compute the size and order of EAN. Note that the addition of each new node leads to $d+1$ new $(d+1)$ cliques and $d+1$ new edges. Then, at step 1, we add expected $L_v(1)=(d+2)q$ new nodes and $L_e(1)=(d+2)(d+1)q$ new edges to the graph. After simple calculations, one can obtain that at $t_i(t_i>1)$ the numbers of newly born nodes and edges are $L_v(t_i)=q(d+2)(1+dq)^{t_i-1}$ and $L_e(t_i)=q(d+1)(d+2)(1+dq)^{t_i-1}$, respectively. Thus the average number of total nodes N_t and edges E_t present at step t is

$$N_t=(d+2)+\sum_{t_i=1}^{t}L_v(t_i)=(d+2)\frac{(1+dq)^t+d-1}{d} \quad (1)$$

and

$$E_t=\frac{(d+2)(d+1)}{2}+\sum_{t_i=1}^{t}L_e(t_i)=(d+2)(d+1)\frac{2(1+dq)^t+d-2}{2d}, \quad (2)$$

respectively. So for large t, The average degree $\bar{k}_t=\frac{2E_t}{N_t}$ is approximately $2(d+1)$. Moreover, when $d=2$, we have $E_t=3N_t-6$. Thus in this case, all networks are maximal planar networks (or graphs) [42].

B. General deterministic Apollonian networks

According to the construction process of d-dimensional Apollonian packings, in Ref. [41], a generation algorithm for d-dimensional Apollonian networks was proposed. Here we generalize the algorithm to establish general deterministic Apollonian networks (GDANs). The network, denoted by $G(d,t)$ after t generations with $d\geqslant 2$ and $t\geqslant 0$, is constructed in an iterative way. For $t=0$, $G(d,0)$ is a complete graph K_{d+2} [or $(d+2)$ clique]. For $t\geqslant 1$, $G(d,t)$ is obtained from $G(d,t-1)$. For each of the existing subgraphs of $G(d,t-1)$ that is isomorphic to a $(d+1)$ clique and created at step $t-1$, m new vertices are created, and each is connected to all the vertices of this subgraph. In the limit of infinite generations we obtain GDAN. When $m=1$, the networks are reduced to DANs [24,25,41].

Let $L_v(t)$, $L_e(t)$, and $K_{(d+1),t}$ be the numbers of vertices, edges, and $(d+1)$ cliques created at step t, respectively. Because the addition of each new vertex leads to $d+1$ new $(d+1)$ cliques and $d+1$ new edges, we have $L_v(t)=mK_{(d+1),t-1}$, $L_e(t)=(d+1)L_v(t)$, and $K_{(d+1),t}=(d+1)L_v(t)$. Thus one can easily obtain $K_{(d+1),t}=m(d+1)K_{(d+1),t-1}=(d+2)[m(d+1)]^t$ $(t\geqslant 0)$, $L_v(t)=(d+2)m^t(d+1)^{t-1}$ $(t>0)$, and $L_e(t)=(d+2)m^t(d+1)^t$ $(t>0)$. From these results, we can easily compute the size and order of the networks. The total number of vertices N_t and edges E_t present at step t is

$$N_t=\sum_{t_i=0}^{t}n_v(t_i)=\frac{m(d+2)[m^t(d+1)^t-1]}{m(d+1)-1}+d+2 \quad (3)$$

and

$$E_t=\sum_{t_i=0}^{t}n_e(t_i)=\frac{m(d+2)(d+1)[m^t(d+1)^t-1]}{m(d+1)-1}+\frac{(d+2)(d+1)}{2}, \quad (4)$$

respectively. So for large large, The average degree $\bar{k}_t=\frac{2E_t}{N_t}$ approaches $2(d+1)$.

IV. RELEVANT CHARACTERISTICS OF THE NETWORKS

In the following we will study the topology properties of EAN and GDAN, in terms of the degree distribution, clustering coefficient, average path length, and diameter.

A. Degree distribution

The degree distribution is one of the most important statistical characteristics of a network. We will discuss the degree distribution of EAN and GDAN in detail.

1. Evolutionary Apollonian networks

When a new node i is added to the graph at step t_i, it has degree $d+1$ and forms $d+1$ active $(d+1)$ cliques. Let $L_c(i,t)$ be the number of active $(d+1)$ cliques at step t that will possibly created new nodes connected to the node i at step $t+1$. Then at step t_i, $L_c(i,t_i)=d+1$. At step t_i+1, there are $(d+1)q$ new nodes which forms $(d+1)qd$ new active $(d+1)$ cliques containing i, and there are $(d+1)q$ active $(d+1)$ cliques of i are deactivated at the same time. Thus $L_c(i,t_i+1)=(d+1)[1+(d-1)q]$. It is not difficult to find the following relation: $L_c(i,t+1)=L_c(i,t)[1+(d-1)q]$. Since $L_c(i,t_i)=d+1$, we have $L_c(i,t)=(d+1)[1+(d-1)q]^{t-t_i}$. Then the degree $k_i(t)$ of node i at time t is

$$k_i(t)=d+1+q\sum_{\tau=t_i}^{t-1}L_c(i,\tau)=(d+1)\left(\frac{[1+(d-1)q]^{t-t_i}+d-2}{d-1}\right). \quad (5)$$

Since the degree of each node has been obtained explicitly as in Eq. (5), we can get the degree distribution via its cumulative distribution, i.e., $P_{\mathrm{cum}}(k)\equiv\Sigma_{k'\geqslant k}N(k',t)/N_t\sim k^{1-\gamma}$, where $N(k',t)$ denotes the number of nodes with degree k'. The detailed analysis is given as follows. For a degree k

$$k=(d+1)\left(\frac{[1+(d-1)q]^{t-m}+d-2}{d-1}\right),$$

there are $L_v(m)=q(d+2)(1+dq)^{m-1}$ nodes with this exact degree, all of which were born at step m. All nodes born at time m or earlier have this or a higher degree. So we have

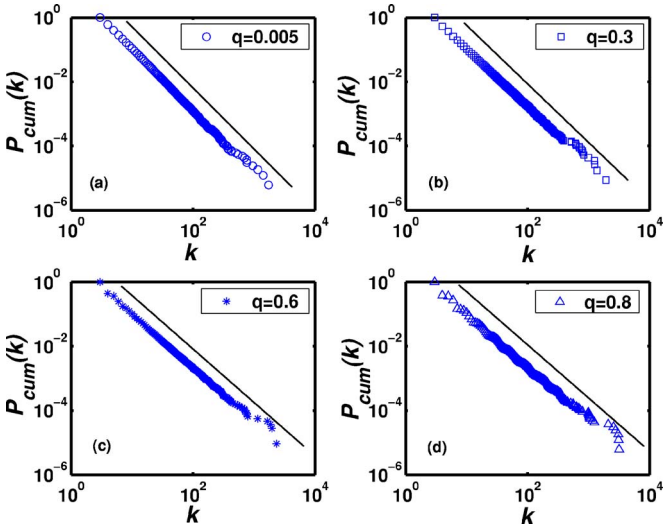

FIG. 3. (Color online) The cumulative degree distribution $P_{\mathrm{cum}}(k)$ at various q values for dimension $d=2$. The circles (a), squares (b), stars (c), and triangles (d) denote the simulation results for networks with different evolutionary steps $t=1110$, $t=23$, $t=14$, and $t=12$, respectively. The four straight lines are the theoretical results of $\gamma(d,q)$ as provided by Eq. (6). All data are from the average of ten independent simulations.

$$\sum_{k' \geqslant k} N(k',t) = (d+2) + \sum_{s=1}^{m} L_v(s) = (d+2)\frac{(1+dq)^m + d - 1}{d}.$$

As the total number of nodes at step t is given in Eq. (1) we have

$$\left[(d+1)\left(\frac{[1+(d-1)q]^{t-m}+d-2}{d-1}\right)\right]^{1-\gamma} = \frac{(d+2)\dfrac{(1+dq)^m+d-1}{d}}{(d+2)\dfrac{(1+dq)^t+d-1}{d}}.$$

Therefore, for large t we obtain

$$[[1+(d-1)q]^{t-m}]^{1-\gamma} = (1+dq)^{m-t}$$

and

$$\gamma \approx 1 + \frac{\ln(1+dq)}{\ln[1+(d-1)q]}. \tag{6}$$

Thus, the degree exponent γ is a continuous function of d and q, and belongs to the interval (2,3]. For any fixed d, as q decrease from 1 to 0, γ increases from $1+\frac{\ln(1+d)}{\ln d}$ to $2+\frac{1}{d-1}$. In the case $d=2$, γ can be tunable between 2.58496 and 3. In the two limitations, i.e., $q=1$ and $q\to 0$ (but $q\neq 0$), the evolutionary Apollonian network reduces to the deterministic Apollonian networks [24,25,41] and their stochastic variants [42,43], respectively. Figure 3 shows, on a logarithmic scale, the scaling behavior of the cumulative degree distribution $P_{\mathrm{cum}}(k)$ for different values of q in the case of $d=2$. Simulation results agree very well with the analytical ones.

2. General deterministic Apollonian networks

Considering a vertex i added to the networks at step t_i. Let $K_{(d+1)}(i,t)$ be the number of newly created $(d+1)$ cliques at step t containing vertex i. These new cliques will create new vertices connected to the vertex i at step $t+1$. At step t_i, $K_{(d+1)}(i,t_i)=d+1$. From the iterative process, one can see that each new neighbor of i generates d new $(d+1)$ cliques with i belonging to them. Then it is not difficult to find the following relations:

$$\Delta k_i(t) = k_i(t) - k_i(t-1) = mK_{(d+1)}(i,t-1)$$

and

$$K_{(d+1)}(i,t) = dmK_{(d+1)}(i,t-1) = (d+1)(dm)^{t-t_i}.$$

Then the degree of vertex i becomes

$$k_i(t) = k_i(t_i) + m\sum_{\tau=t_i}^{t-1} K_{(d+1)}(i,\tau) = \frac{m(d+1)[(md)^{t-t_i}-1]+d^2-1}{d-1}. \tag{7}$$

For a degree k

$$k = \frac{m(d+1)[(md)^{t-p}-1]+d^2-1}{d-1},$$

there are $n_v(p)=(d+2)m^p(d+1)^{p-1}$ vertices with this exact degree. Also, we have

$$\sum_{k' \geqslant k} N(k',t) = \sum_{s=0}^{p} n_v(s) = \frac{m(d+2)[m^p(d+1)^p-1]}{m(d+1)-1} + d + 2. \tag{8}$$

From the definition of cumulative degree distribution, we have

$$\left[\frac{m(d+1)[(md)^{t-p}-1]+d^2-1}{d-1}\right]^{1-\gamma} = \frac{\dfrac{m(d+2)[m^p(d+1)^p-1]}{m(d+1)-1}+d+2}{\dfrac{m(d+2)[m^t(d+1)^t-1]}{m(d+1)-1}+d+2}.$$

When t is large enough, one can obtain

$$[(md)^{t-p}]^{1-\gamma} = [m(d+1)]^{p-t}$$

and

$$\gamma \approx 1 + \frac{\ln[m(d+1)]}{\ln(md)}. \tag{9}$$

For $m=1$, Eq. (9) recovers the results previously obtained in Refs. [24,25,41].

B. Clustering coefficient

The clustering coefficient [6] C_i of node i is defined as the ratio between the number of edges e_i that actually exist

among the k_i neighbors of node i and its maximum possible value $k_i(k_i-1)/2$, i.e., $C_i=2e_i/[k_i(k_i-1)]$. The clustering coefficient of the whole network is the average of $C_i's$ over all nodes in the network.

For both EAN and GDAN, the analytical expression of clustering coefficient $C(k)$ for a single node with degree k can be derived exactly. When a node is created it is connected to all the nodes of a $(d+1)$ clique, in which nodes are completely interconnected. So its degree and clustering coefficient are $d+1$ and 1, respectively. In the following steps, if its degree increases one by a newly created node connecting to it, then there must be d existing neighbors of it attaching to the new node at the same time. Thus for a node of degree k, we have

$$C(k)=\frac{\frac{d(d+1)}{2}+d(k-d-1)}{\frac{k(k-1)}{2}}=\frac{2d\left(k-\frac{d+1}{2}\right)}{k(k-1)}, \quad (10)$$

which depends on degree k and dimension d. For $k \gg d$, the $C(k)$ is inversely proportional to degree k. The scaling $C(k)\sim k^{-1}$ has been found for some network models [24,25,28,30–32,41–43], including DANs and RANs [24,25,41–43], and has also been observed in several real-life networks [30].

Using Eq. (10), we can obtain the clustering $\bar{C}_t$ of the networks at step t:

$$\bar{C}_t=\frac{1}{N_t}\sum_{r=0}^{t}\frac{2d\left(D_r-\frac{d+1}{2}\right)L_v(r)}{D_r(D_r-1)}, \quad (11)$$

where the sum runs over all the nodes and D_r is the degree of the nodes created at step r, which is given by Eq. (5) or (7).

1. Evolutionary Apollonian networks

For EAN, it can be easily proved that for any fixed d, $\bar{C}_t$ increases with q, and that for arbitrary fixed q, $\bar{C}_t$ increases with d. Exactly analytical computation shows: in the case $d=2$, when q increases from 0 to 1, $\bar{C}$ grows from 0.7366 to 0.8284, with a special value $\bar{C}_t=0.7934$ for $q=0.5$. Likewise, in the case $d=3$, $\bar{C}$ increases from 0.8021 to 0.8852 when q increases from 0 to 1, especially $\bar{C}_t=0.8585$ for $q=0.5$ (see also Refs. [24,41–43]). Therefore, the evolutionary networks are highly clustered. Figure 4 shows the clustering coefficient of the network as a function of q for $d=2$ and $d=3$, respectively, which is in accordance with our above conclusions. From Figs. 3 and 4, one can see that both degree exponent γ and clustering coefficient $\bar{C}_t$ depend on the parameter q. The mechanism resulting in this relation deserves further study. The fact that a biased choice of the cliques at each iteration may be a possible explanation, see Ref. [47].

2. General deterministic Apollonian networks

For GDANs, in the infinite network order limit $(N_t\to\infty)$, Eq. (11) converges to a nonzero value. When $d=2$, for

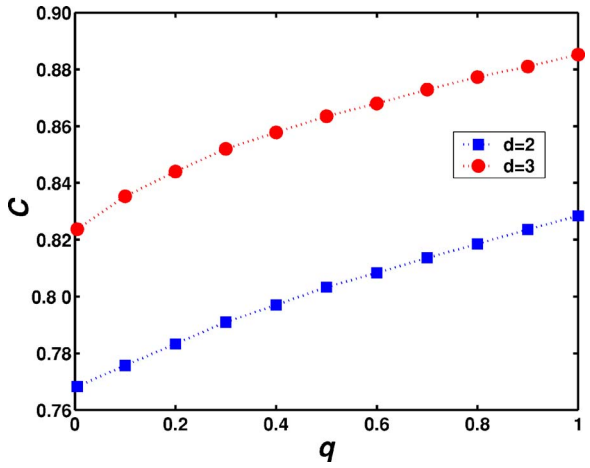

FIG. 4. (Color online) The clustering coefficient of the whole network as a function of q and d. Results are averaged over ten network realizations for each datum.

$m=1$, 2, and 3, C is equal to 0.8284, 0.8602, and 0.8972, respectively. When $m=2$, for $d=2$, 3, and 4, C are 0.8602, 0.9017, and 0.9244, respectively. Therefore, the clustering coefficient of GDANs is very high. Moreover, similarly to the degree exponent γ, clustering coefficient C is determined by both d and m. Figure 5 shows the dependence of C on d and m. From Figs. 5(a) and 5(b), one can see that for any fixed m, C increases with d. But the dependence relation of C on m [see Fig. 5(b)] is more complex: (i) when $m\leqslant 2$ and $d\leqslant 6$, for the same d, C increases with m; (ii) when $m\leqslant 2$ and $d>6$, for the same d, C decreases with m; (iii) when $m\geqslant 3$, for arbitrary fixed d, C increases with m. Further effort should be extended to this complicated relation.

C. Average path length for evolutionary Apollonian networks

We label the nodes by their creation times $v=1,2,3,\ldots,N-1,N$. Using $l(N)$ to represent the APL of the our networks with order N, then we have $l(N)=\frac{2\sigma(N)}{N(N-1)}$, where $\sigma(N)=\Sigma_{1\leqslant i<j\leqslant N}d_{i,j}$ is the total distance, in which $d_{i,j}$

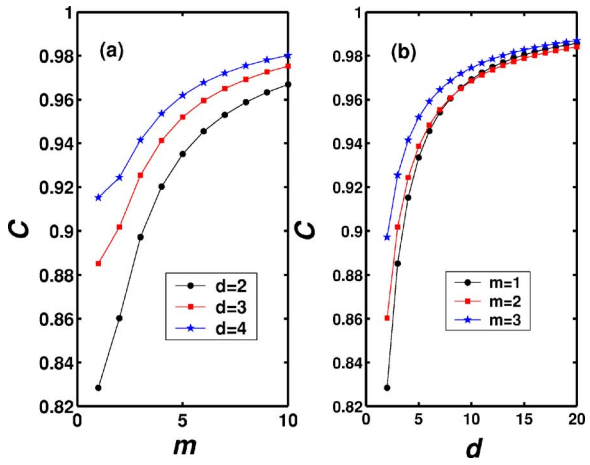

FIG. 5. (Color online) The dependence relation of C on d and m.

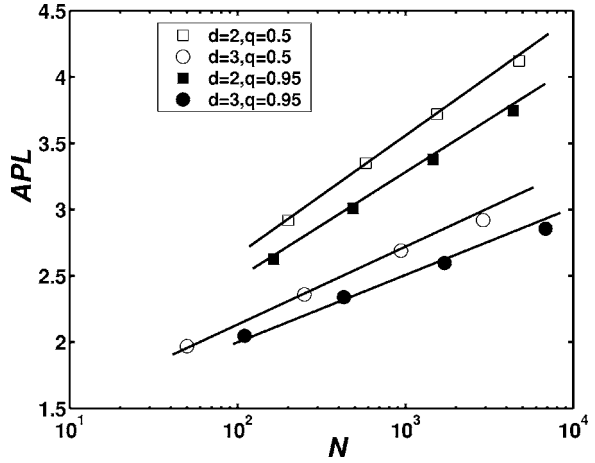

FIG. 6. Semilogarithmic graph of the APL vs the network order N. In addition to N, APL depends on d and q. Each data is obtained by ten independent network realizations. The lines are linear functions of $\ln N$.

is the smallest distance between node i with node j. Now we study the APL of the present model by using the approach similar to that in Refs. [42,43].

Since the distances between existing node pairs will not be affected by the addition of new nodes, thus we have

$$\sigma(N+1) = \sigma(N) + \sum_{i=1}^{N} d_{i,N+1}. \tag{12}$$

As in the analysis of Refs. [42,43], Eq. (12) can be rewritten approximately as

$$\sigma(N+1) = \sigma(N) + N + (N-d-1)l(N-d). \tag{13}$$

It is clear that

$$(N-d-1)l(N-d) = \frac{2\sigma(N-d)}{N-d} < \frac{2\sigma(N)}{N}. \tag{14}$$

From Eqs. (13) and (14), we can provide an upper bound for the variation of $\sigma(N)$ as

$$\frac{d\sigma(N)}{dN} = N + \frac{2\sigma(N)}{N}, \tag{15}$$

which leads to

$$\sigma(N) = N^2(\ln N + \beta), \tag{16}$$

where β is a constant. As $\sigma(N) \sim N^2 \ln N$, we have $l(N) \sim \ln N$.

It should be emphasized that as Eq. (15) has been deduced from an inequality, then $l(N)$ increases at most as $\ln N$ with N. Here we only give an upper bound for APL, which increases slightly slower than $\ln N$. Thus, our model exhibits the small-world property. Especially, in the case of $q=1$, we can exactly compute the diameter of the network, which is the maximum distance between all node pairs of a graph. A previously reported analytical result has shown that the diameter grows logarithmically with the order of the network [41]. In Fig. 6, we report the simulation results on the APL of networks for different q and d. From Fig. 6, one can see that for fixed d, APL decreases with increasing q; and for fixed q, APL is a decreasing function of d. When network order N is small, APL is a linear function of $\ln N$; while N becomes large, APL increases slightly slower than $\ln N$.

D. Diameter for general deterministic Apollonian networks

The diameter of a network characterizes the maximum communication delay in the network and is defined as the longest shortest path between all pairs of vertices. In what follows, the notations $\lceil x \rceil$ and $\lfloor x \rfloor$ represent the integers obtained by rounding x to the nearest integers towards infinity and minus infinity, respectively. Now we compute the diameter of $A(d,t)$, denoted by $\mathrm{diam}[A(d,t)]$ for $d \geqslant 2$:

Step 0. The diameter is 1.

Steps 1 to $\lceil \frac{d}{2} \rceil$. In this case, the diameter is 2, since any new vertex is by construction connected to a $(d+1)$ clique, and since any $(d+1)$ clique during those steps contains at least $\frac{d}{2}+2$ (d even) or $\frac{d+1}{2}+1$ (d odd) vertices from the initial $(d+2)$ clique $A(d,0)$ obtained after step 0. Hence, any two newly added vertices u and v will be connected respectively to sets S_u and S_v, with $S_u \subseteq V[A(d,0)]$ and $S_v \subseteq V[A(d,0)]$, where $V[A(d,0)]$ is the vertex set of $A(d,0)$; however, since $|S_u| \geqslant \frac{d}{2}+2$ (d even) and $|S_v| \geqslant \frac{d+1}{2}+1$ (d odd), where $|S|$ denotes the number of elements in set S, we conclude that $S_u \cap S_v \neq \emptyset$, and thus the diameter is 2.

Steps $\lceil \frac{d}{2} \rceil + 1$ *to* $d+1$. In any of those steps, some newly added vertices might not share a neighbor in the original $(d+2)$ clique $A(d,0)$ obtained after step 0; however, any newly added vertex is connected to at least one vertex of the initial clique $A(d,0)$. Thus, the diameter equals to 3.

Further steps. Clearly, at any step $t \geqslant d+2$, the diameter always lies between a pair of vertices that have just been created at this step. We will call the newly created vertices "outer" vertices. At any step $t \geqslant d+2$, we note that an outer vertex cannot be connected with two or more vertices that were created during the same step $0 < t' \leqslant t-1$. Moreover, by construction no two vertices that were created during a given step are neighbors, thus they cannot be part of the same $(d+1)$ clique. Thus, for any step $t \geqslant d+2$, some outer vertices are connected with vertices that appeared at pairwise different steps. Thus, there exists an outer vertex v_t created at step t, which is connected to vertices $v_i's$, $1 \leqslant i \leqslant t-1$, all of which are pairwise distinct. We conclude that v_t is necessarily connected to a vertex that was created at a step $t_0 \leqslant t-d-1$. If we repeat this argument, then we obtain an upper bound on the distance from v_t to the initial clique $A(d,0)$. Let $t=\alpha(d+1)+\beta$, where $1 \leqslant \beta \leqslant d+1$. Then, we see that v_t is at distance at most $\alpha+1$ from a vertex in $A(d,0)$. Hence any two vertices v_t and w_t in $A(d,t)$ lie at distance at most $2(\alpha+1)+1$; however, depending on β, this distance can be reduced by 1, since when $\beta \leqslant \lceil \frac{d}{2} \rceil$, we know that two vertices created at step β share at least a neighbor in $A(d,0)$. Thus, when $1 \leqslant p \leqslant \lceil \frac{d}{2} \rceil$, $\mathrm{diam}[A(d,t)] \leqslant 2(\alpha+1)$, while when $\lceil \frac{d}{2} \rceil + 1 \leqslant p \leqslant d+1$, $\mathrm{diam}[A(d,t)] \leqslant 2(\alpha+1)+1$. One can see that these distance bounds can be reached by pairs of outer vertices created at step t. More precisely, those two vertices

v_t and w_t share the property that they are connected to d vertices that appeared respectively at steps $t-1, t-2, \ldots, t-d-1$.

Based on the above arguments, one can easily see that for $t > d+2$, the diameter increases by 2 every $d+1$ steps. More precisely, we have the following result, for any $d \geqslant 1$ and $t \geqslant 1$ (when $t=0$, the diameter is clearly equal to 1):

$$\mathrm{diam}[A(d,t)] = 2\left(\left\lfloor \frac{t-1}{d+1} \right\rfloor + 1\right) + f(d,t),$$

where $f(d,t)=0$ if $t-\lfloor \frac{t-1}{d+1} \rfloor (d+1) \lceil \rangle \frac{d}{2}$, and 1 otherwise.

In the limit of large t, $\mathrm{diam}[A(d,t)] \sim \frac{2t}{d+1}$, while $N_t \sim [m(d+1)]^t$, thus the diameter is small and scales logarithmically with the network order.

V. SYNCHRONIZATION ON SOME LIMITING CASES FOR EVOLUTIONARY APOLLONIAN NETWORKS

The ultimate goal of the study of network structure is to study and understand the workings of systems built upon those networks [1–4]. Recently, some researchers have focused on the analysis of functional or dynamical aspects of processes occurring on networks. One particular issue attracting much attention is the synchronizability of oscillator coupling networks [48]. Synchronization is observed in diverse natural and man-made systems and is directly related to many specific problems in a variety of different disciplines. It has found practical applications in many fields including communications, optics, neural networks, and geophysics [49–54]. After studying the relevant characteristics of network structure, which is described in the last sections, we will study the synchronization behavior on the networks.

We follow the general framework proposed in Refs. [55,56], where a criterion based on spectral techniques was established to determine the stability of synchronized states on networks. Consider a network of N identical dynamical systems with linearly and symmetric coupling between oscillators. The set of equations of motion for the system are

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j), \tag{17}$$

where $\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i)$ governs the dynamics of each individual node, $\mathbf{H}(\mathbf{x}_j)$ is the output function and σ the coupling strength, and G_{ij} is the Laplacian matrix, defined by $G_{ii}=k_i$ if the degree of node i is k_i, $G_{ij}=-1$ if nodes i and j are connected, and $G_{ij}=0$ otherwise.

Since matrix G is positive semidefinite and each rows of it has zero sum, all eigenvalues of G are real and non-negative and the smallest one is always equal to zero. We order the eigenvalues as $0=\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$. Then one can use the ratio of the maximum eigenvalue λ_N to the smallest nonzero one λ_2 to measure the synchronizability of the network [55,56]. If the eigenratio $R=\lambda_N/\lambda_2$ satisfies $R < \alpha_2/\alpha_1$, we say the network is synchronizable. Here the eigenratio R depends on the the network topology, while α_2/α_1 depends exclusively on the dynamics of individual oscillator and the output function. Ratio $R=\lambda_N/\lambda_2$ represents the synchronizability of the network: the larger the ratio, the more difficult it is to synchronize the oscillators, and vice versa.

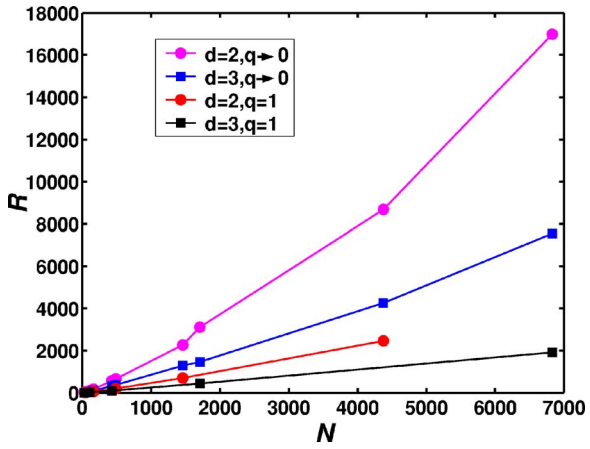

FIG. 7. (Color online) The eigenratio R as a function of network order N. All quantities for random networks are averaged over 50 realizations.

After reducing the issue of synchronizability to finding eigenvalues of the Laplacian matrix G, we now investigate the synchronization of our networks. Here we only study two limiting cases: $q=1$ [24,25,41] and $q \to 0$ (but $q \neq 0$) [42,43]. The eigenratio R of different networks is obtained numerically for different d and q, as exhibited in Fig. 7. One can see that for the same d and N, R of the deterministic networks is smaller than that of their corresponding random versions, which implies that the synchronizability of the former is better. Even in the case of different d, the deterministic networks ($d=2$) are easier to synchronize than those random networks ($d=3$) with the same order but different average node degree.

Why coupling systems on the two class of networks exhibit very different synchronizability? Previously reported results have indicated that underlying network structures play significant roles in the synchronizability of coupled oscillators. However, the key structural feature that determines the collective synchronization behavior remains unclear. Many works have discussed this issue. Some authors believe that a shorter APL tends to enhance synchronization [55,57,58]. In contrast, Nishikawa *et al.* reported that synchronizability is suppressed as the degree distribution becomes more heterogeneous, even for a shorter APL [59]. Also, some investigations showed that R decreases whenever the betweeness heterogeneity decreases [60], while an opposite conclusion was claimed in Ref. [61]. In Ref. [62], the authors asserted that larger average node degree corresponds to better synchroizability. All these may rationally explain the relations between synchronizability and network structure in some cases, but do not well account for the synchronizability on the evolutionary Apollonian networks. More recently, some other authors have presented that structure and distribution of hubs is the key to what enhances synchronizability [63,64], which may be a possible explanation for the better synchronizability of deterministic networks when compared to random ones. But we speculate that the synchronizability on the evolutionary Apollonian networks is

not determined by a single structure property, but by the combination of APL, heterogeneity of degree distribution, betweeness centrality, modularity, mean node degree, and so on [63], which need further research.

VI. CONCLUSION AND DISCUSSION

In summary, on the basis of Apollonian packings, we have proposed and studied two kinds of evolving networks: evolutionary Apollonian networks (EANs) and general deterministic Apollonian networks (GDANs). According to the network construction processes we have presented two algorithms to generate the networks, based on which we have obtained the analytical and numerical results for degree distribution and clustering coefficient, as well as the average path length, which agree well with a large amount of real observations. The degree exponents can be adjusted continuously between 2 and 3, and the clustering coefficient is very large. Moreover, we have studied the synchronization of some limiting cases of the EAN and found that the stochastic networks are more difficult to synchronize than their deterministic counterparts.

Because of their three important properties: power-law degree distribution, small intervertex separation, and large clustering coefficient, the proposed networks possess good structural features in accordance with a variety of real-life networks. For the special case of $d=2$, the networks are maximal planar graphs. This may be helpful for designing printed circuits. Moreover, our networks consist of complete graphs, which have been observed in a variety of the real-world networks, such as movie actor collaboration networks, scientific collaboration networks and networks of company directors [1–4].

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